1 Linear Equations in Linear Algebra

1.1 SOLUTIONS

Notes: The key exercises are 7 (or 11 or 12), 19–22, and 25. For brevity, the symbols R1, R2,…, stand for row 1 (or equation 1), row 2 (or equation 2), and so on. Additional notes are at the end of the section.

1. $\begin{aligned} x_1 + 5x_2 &= 7 \\ -2x_1 - 7x_2 &= -5 \end{aligned}$ $\quad \begin{bmatrix} 1 & 5 & 7 \\ -2 & -7 & -5 \end{bmatrix}$

Replace R2 by R2 + (2)R1 and obtain: $\begin{aligned} x_1 + 5x_2 &= 7 \\ 3x_2 &= 9 \end{aligned}$ $\quad \begin{bmatrix} 1 & 5 & 7 \\ 0 & 3 & 9 \end{bmatrix}$

Scale R2 by 1/3: $\begin{aligned} x_1 + 5x_2 &= 7 \\ x_2 &= 3 \end{aligned}$ $\quad \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 3 \end{bmatrix}$

Replace R1 by R1 + (–5)R2: $\begin{aligned} x_1 \quad &= -8 \\ x_2 &= 3 \end{aligned}$ $\quad \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 3 \end{bmatrix}$

The solution is (x_1, x_2) = (–8, 3), or simply (–8, 3).

2. $\begin{aligned} 2x_1 + 4x_2 &= -4 \\ 5x_1 + 7x_2 &= 11 \end{aligned}$ $\quad \begin{bmatrix} 2 & 4 & -4 \\ 5 & 7 & 11 \end{bmatrix}$

Scale R1 by 1/2 and obtain: $\begin{aligned} x_1 + 2x_2 &= -2 \\ 5x_1 + 7x_2 &= 11 \end{aligned}$ $\quad \begin{bmatrix} 1 & 2 & -2 \\ 5 & 7 & 11 \end{bmatrix}$

Replace R2 by R2 + (–5)R1: $\begin{aligned} x_1 + 2x_2 &= -2 \\ -3x_2 &= 21 \end{aligned}$ $\quad \begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 21 \end{bmatrix}$

Scale R2 by –1/3: $\begin{aligned} x_1 + 2x_2 &= -2 \\ x_2 &= -7 \end{aligned}$ $\quad \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -7 \end{bmatrix}$

Replace R1 by R1 + (–2)R2: $\begin{aligned} x_1 \quad &= 12 \\ x_2 &= -7 \end{aligned}$ $\quad \begin{bmatrix} 1 & 0 & 12 \\ 0 & 1 & -7 \end{bmatrix}$

The solution is (x_1, x_2) = (12, –7), or simply (12, –7).

3. The point of intersection satisfies the system of two linear equations:

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ x_1 - 2x_2 &= -2 \end{aligned} \qquad \begin{bmatrix} 1 & 5 & 7 \\ 1 & -2 & -2 \end{bmatrix}$$

Replace R2 by R2 + (–1)R1 and obtain:

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ -7x_2 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & 5 & 7 \\ 0 & -7 & -9 \end{bmatrix}$$

Scale R2 by –1/7:

$$\begin{aligned} x_1 + 5x_2 &= 7 \\ x_2 &= 9/7 \end{aligned} \qquad \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 9/7 \end{bmatrix}$$

Replace R1 by R1 + (–5)R2:

$$\begin{aligned} x_1 \quad &= 4/7 \\ x_2 &= 9/7 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 4/7 \\ 0 & 1 & 9/7 \end{bmatrix}$$

The point of intersection is $(x_1, x_2) = (4/7, 9/7)$.

4. The point of intersection satisfies the system of two linear equations:

$$\begin{aligned} x_1 - 5x_2 &= 1 \\ 3x_1 - 7x_2 &= 5 \end{aligned} \qquad \begin{bmatrix} 1 & -5 & 1 \\ 3 & -7 & 5 \end{bmatrix}$$

Replace R2 by R2 + (–3)R1 and obtain:

$$\begin{aligned} x_1 - 5x_2 &= 1 \\ 8x_2 &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & -5 & 1 \\ 0 & 8 & 2 \end{bmatrix}$$

Scale R2 by 1/8:

$$\begin{aligned} x_1 - 5x_2 &= 1 \\ x_2 &= 1/4 \end{aligned} \qquad \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1/4 \end{bmatrix}$$

Replace R1 by R1 + (5)R2:

$$\begin{aligned} x_1 \quad &= 9/4 \\ x_2 &= 1/4 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 9/4 \\ 0 & 1 & 1/4 \end{bmatrix}$$

The point of intersection is $(x_1, x_2) = (9/4, 1/4)$.

5. The system is already in "triangular" form. The fourth equation is $x_4 = -5$, and the other equations do not contain the variable x_4. The next two steps should be to use the variable x_3 in the third equation to eliminate that variable from the first two equations. In matrix notation, that means to replace R2 by its sum with 3 times R3, and then replace R1 by its sum with –5 times R3.

6. One more step will put the system in triangular form. Replace R4 by its sum with –3 times R3, which produces $\begin{bmatrix} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & -5 & 15 \end{bmatrix}$. After that, the next step is to scale the fourth row by –1/5.

7. Ordinarily, the next step would be to interchange R3 and R4, to put a 1 in the third row and third column. But in this case, the third row of the augmented matrix corresponds to the equation $0\,x_1 + 0\,x_2 + 0\,x_3 = 1$, or simply, $0 = 1$. A system containing this condition has no solution. Further row operations are unnecessary once an equation such as $0 = 1$ is evident.
The solution set is empty.

8. The standard row operations are:

$$\begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The solution set contains one solution: (0, 0, 0).

9. The system has already been reduced to triangular form. Begin by scaling the fourth row by 1/2 and then replacing R3 by R3 + (3)R4:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Next, replace R2 by R2 + (3)R3. Finally, replace R1 by R1 + R2:

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution set contains one solution: (4, 8, 5, 2).

10. The system has already been reduced to triangular form. Use the 1 in the fourth row to change the –4 and 3 above it to zeros. That is, replace R2 by R2 + (4)R4 and replace R1 by R1 + (–3)R4. For the final step, replace R1 by R1 + (2)R2.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & -2 \\ 0 & 1 & 0 & -4 & 7 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

The solution set contains one solution: (–3, –5, 6, –3).

11. First, swap R1 and R2. Then replace R3 by R3 + (–3)R1. Finally, replace R3 by R3 + (2)R2.

$$\begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The system is inconsistent, because the last row would require that 0 = 2 if there were a solution. The solution set is empty.

12. Replace R2 by R2 + (–3)R1 and replace R3 by R3 + (4)R1. Finally, replace R3 by R3 + (3)R2.

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The system is inconsistent, because the last row would require that 0 = 3 if there were a solution. The solution set is empty.

13. Replace R2 by R2 + (–2)R1. Then interchange R2 and R3. Next replace R3 by R3 + (–2)R2. Then divide R3 by 5. Finally, replace R1 by R1 + (–2)R3.

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$. The solution is (5, 3, –1).

14. Replace R2 by R2 + R1. Then interchange R2 and R3. Next replace R3 by R3 + 2R2. Then divide R3 by 7. Next replace R2 by R2 + (–1)R3. Finally, replace R1 by R1 + 3R2.

$$\begin{bmatrix} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$. The solution is (2, –1, 1).

15. First, replace R4 by R4 + (–3)R1, then replace R3 by R3 + (2)R2, and finally replace R4 by R4 + (3)R3.

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix}.$$

The resulting triangular system indicates that a solution exists. In fact, using the argument from Example 2, one can see that the solution is unique.

16. First replace R4 by R4 + (2)R1 and replace R4 by R4 + (–3/2)R2. (One could also scale R2 before adding to R4, but the arithmetic is rather easy keeping R2 unchanged.) Finally, replace R4 by R4 + R3.

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is now in triangular form and has a solution. The next section discusses how to continue with this type of system.