The problems in this chapter are primarily mathematical. They are intended to give students some practice with the concepts introduced in Chapter 2, but the problems in themselves offer few economic insights. Consequently, no commentary is provided. Results from some of the analytical problems are used in later chapters, however, and in those cases the student will be directed back to this chapter.

**CHAPTER 2:**

**Mathematics for Microeconomics**

**Solutions**

**2.1** 

 a.  

 b. $U\_{x}=8, U\_{y}=12$Constraining  creates an implicit function between the variables. The slope of this function is given by  for combinations of *x* and *y* that satisfy the constraint.

c. Since , we know that at this point .

d. The  contour line is an ellipse centered at the origin. The slope of the line at any point is given by  Notice that this slope becomes more negative as *x* increases and *y* decreases.

**2.2** a. Profits are given by  The maximum value is found by setting the derivative equal to 0:

 ,

 implies  and 

 b. Since  this is a global maximum.

 c.   So,  obeys 

**2.3** First, use the substitution method. Substituting  yields  Taking the first-order condition,  and solving yields , and  Since  this is a local and global maximum.

Next, use the Lagrange method. The Lagrangian is  The first-order conditions are

 

Solving simultaneously,  Using the constraint gives  and 

**2.4** Setting up the Lagrangian,  The first-order conditions are

 

So  Using the constraint  gives  and Note that the solution is the same here as in Problem 2.3, but here the value for the Lagrangian multiplier is the reciprocal of the value in Problem 2.3.

**2.5** a. The height of the ball is given by  The value of  for which height is maximized is found by using the first-order condition:  implying 

 b. Substituting for 

 

 Hence,

 

c. Differentiation of the original function at its optimal value yields

 

 Because the optimal value of  depends on 

 

as was also shown in part (c).

d. If  Maximum height is  If  maximum height is  a reduction of 0.08. This could have been predicted from the envelope theorem, since

 

**2.6** a. This is the volume of a rectangular solid made from a piece of metal, which is  by  with the defined corner squares removed.

 b. The first-order condition for maximum volume is given by

 

 Applying the quadratic formula to this expression yields

 

 The second value given by the quadratic  is obviously extraneous.

 c. If  

 So volume increases without limit.

 d. This would require a solution using the Lagrangian method. The optimal solution requires solving three nonlinear simultaneous equations, a task not undertaken here. But it seems clear that the solution would involve a different relationship between  and  than in parts (a–c).

**2.7** a. Set up the Lagrangian: The first-order conditions are

 

 Hence,  With  the optimal solution is 

 b. With  solving the first-order conditions yields  and 

 c. If all variables must be nonnegative, it is clear that any positive value for  reduces  Hence, the optimal solution is  and 

d. If  optimal solution is  and  Because  provides a diminishing marginal increment to  as its value increases, whereas  does not, all optimal solutions require that once  reaches 5, any extra amounts be devoted entirely to  In consumer theory, this function can be used to illustrate how diminishing marginal usefulness can be modeled in a very simple setting.

**2.8** a. Because  is the derivative of   is an antiderivative of  By the fundamental theorem of calculus,



where  is the fixed cost, which we will denote  for short. Rearranging,



b. For profit maximization,  implying  But  implies  Profit are

 

If the firm is just breaking even, profit equals 0, implying fixed cost is 

 c. When  and  follow the same steps as in part (b), substituting fixed cost  Profit are

 

 d. Assuming profit maximization, we have



e.

i. Using the above equation, 

ii. The envelope theorem states that  That is, the derivative of the profit function yields this firm’s supply function. Integrating over  shows the change in profits by the fundamental theorem of calculus:

 

**Analytical Problems**

**2.9 Concave and quasi-concave**  **functions**

The proof is most easily accomplished through the use of the matrix algebra of quadratic forms. See, for example, Mas Colell *et al*.,1995, pp. 937–939. Intuitively, because concave functions lie below any tangent plane, their level curves must also be convex. But the converse is not true. Quasi-concave functions may exhibit “increasing returns to scale”; even though their level curves are convex, they may rise above the tangent plane when all variables are increased together.

A counter example would be the Cobb–Douglas function, which is always quasi-concave, but convex when 

**2.10** **The Cobb–Douglas function**

 a. 

Clearly, all the terms in Equation 2.114 are negative.

b. A contour line is found by setting the function equal to a constant: implying  Hence,



Further,



implying the countour line is convex.

c. Using Equation 2.98,  which is negative for 

**2.11** **The power function**

 a. Since  and  the function is concave.

 b. Because  and  Equation 2.98 is satisfied, and the function is concave. Because  Equation 2.114 is also satisfied, so the function is quasi-concave.

 c.  is quasi-concave as is  However,  is not concave for  This can be shown most easily by 

**2.12 Proof of envelope theorem**

a. The Lagrangian for this problem is

 ****

 The first-order conditions are

 

 b., c. Multiplication of each first-order condition by the appropriate deriviative yields

 

 d. The optimal value of  is given by  Differentiation of this with respect to  shows how this optimal value changes with :

 

e. Differentiation of the constraint  yields

 

 f. Multiplying the results from part (e) by  and using parts (b) and (c) yields

 

 This proves the envelope theorem.

g. In Example 2.8, we showed that  This shows how much an extra unit of perimeter would raise the enclosed area. Direct differentiation of the original Lagrangian shows also that

 

This shows that the Lagrange multiplier does indeed show this incremental gain in this problem.

**2.13** **Taylor approximations**

a. A function in one variable is concave if  Using the quadratic Taylor formula to approximate this function at point :



The inequality holds because  But the right-hand side of this equation is the equation for the tangent to the function at point  So we have shown that any concave function must lie on or below the tangent to the function at that point.

 b. A function in two variables is concave if 

 Hence, the quadratic form  will also be negative. But this says that the final portion of the Taylor expansion will be negative (by setting  and ), and hence the function will be below its tangent plane.

**2.14** **More on expected value**

a. The tangent to  at the point  will have the form  for all values of  and  But, because the line  is above the function , we know

 

 This proves Jensen’s inequality.

 b. Use the same procedure as in part (a), but reverse the inequalities.

c. Let  and 

 

d. Use the hint to break up the integral defining expected value:



 e. 1. Show that this function integrates to 1:

 

 2. Calculate the cumulative distribution function:

 

 3. Using the result from part (c):

 

4. To show Markov’s inequality use

 

f. 1. Show that the PDF integrates to 1:



 2. Calculate the expected value:



 3. Calculate ):



 4. All we must do is adjust the PDF so that it now sums to 1 over the new, smaller interval. Since 



 5. The expected value is again found through integration:

 

6. Eliminating the lowest values of *x* increases the expected value of the remaining values.

**2.15** **More on variances**

 a. This is just an application of the definition of variance:

 

b. Here, we let  and apply Markov’s inequality to  and remember that  can only take on positive values.



c. Let   be  independent random variables each with expected value  and variance 





Now, let 





d. Let  and 

 



Hence, variance is minimized for  In this case,  If  (not much of an increase).

e. Suppose that  and  Now







For example, if  then , and optimal diversification requires that the lower risk asset constitute two-thirds of the portfolio. Note, however, that it is still optimal to have some of the higher risk asset because asset returns are independent.

 **2.16** **More on covariances**

 a. This is a direct result of the definition of covariance:

 

b. 

The final line is a result of Problems 2.15a and 2.16a.

c. The presence of the covariance term in the result of Problem 2.16b suggests that the results would differ. In the two-variable case, however, this is not necessarily the situation. For example, suppose that *x* and *y* are identically distributed and that  $Cov(x,y)=rσ^{2}.$Using the prior notation,



The first-order condition for a minimum is



implying



Regardless of the value of  With more than two random variables, however, covariances may indeed affect optimal weightings.

d. If  the correlation coefficient will be either  (if  is positive) or  (if  is negative), since  will factor out of the definition leaving only the ratio of the common variance of the two variables. With less than a perfect linear relationship 

e. If 



Hence,

