SOLUTIONS TO CHAPTER 1

Problem 1.1

(a) Since the growth rate of a variable equals the time derivative of its log, as shown by equation (1.10) in the text, we can write

(1) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln[X(t)Y(t)]}{dt}.$$

Since the log of the product of two variables equals the sum of their logs, we have

(2) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{d[\ln X(t) + \ln Y(t)]}{dt} = \frac{d\ln X(t)}{dt} + \frac{d\ln Y(t)}{dt},$$

or simply

(3) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{X(t)} + \frac{\dot{Y}(t)}{Y(t)}.$$

(b) Again, since the growth rate of a variable equals the time derivative of its log, we can write

(4) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln[X(t)/Y(t)]}{dt}.$$

Since the log of the ratio of two variables equals the difference in their logs, we have

(5) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{d[\ln X(t) - \ln Y(t)]}{dt} = \frac{d\ln X(t)}{dt} - \frac{d\ln Y(t)}{dt},$$

or simply

(6) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{\dot{X}(t)}{X(t)} - \frac{\dot{Y}(t)}{Y(t)}.$$

(c) We have

(7) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{d\ln Z(t)}{dt} = \frac{d\ln[X(t)^{\alpha}]}{dt}.$$

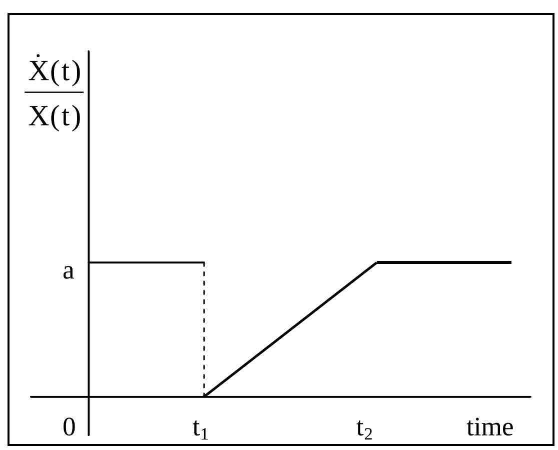
Using the fact that $\ln[X(t)^{\alpha}] = \alpha \ln X(t)$, we have

(8) $$\frac{\dot{Z}(t)}{Z(t)} = \frac{d[\alpha \ln X(t)]}{dt} = \alpha \frac{d\ln X(t)}{dt} = \alpha \frac{\dot{X}(t)}{X(t)},$$

where we have used the fact that α is a constant.

Problem 1.2

(a) Using the information provided in the question, the path of the growth rate of X, $\dot{X}(t)/X(t)$, is depicted in the figure at right.



From time 0 to time t_1, the growth rate of X is constant and equal to a > 0. At time t_1, the growth rate of X drops to 0. From time t_1 to time t_2, the growth rate of X rises gradually from 0 to a. Note that we have made the assumption that $\dot{X}(t)/X(t)$ rises at a constant rate from t_1 to t_2. Finally, after time t_2, the growth rate of X is constant and equal to a again.

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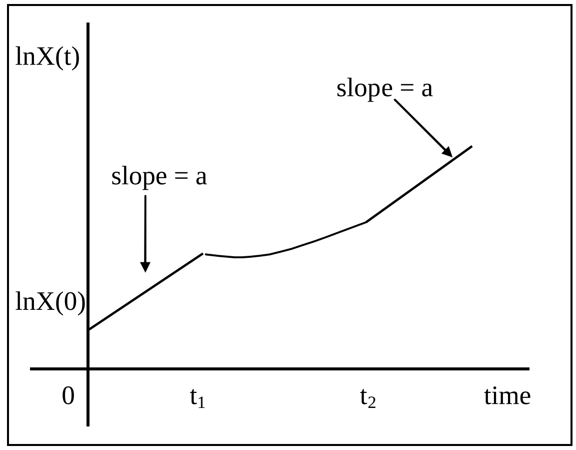
(b) Note that the slope of lnX(t) plotted against time is equal to the growth rate of X(t). That is, we know

$$\frac{d\ln X(t)}{dt} = \frac{\dot{X}(t)}{X(t)}$$

(See equation (1.10) in the text.)

From time 0 to time t_1 the slope of lnX(t) equals a > 0. The lnX(t) locus has an inflection point at t_1, when the growth rate of X(t) changes discontinuously from a to 0. Between t_1 and t_2, the slope of lnX(t) rises gradually from 0 to a. After time t_2 the slope of lnX(t) is constant and equal to a > 0 again.

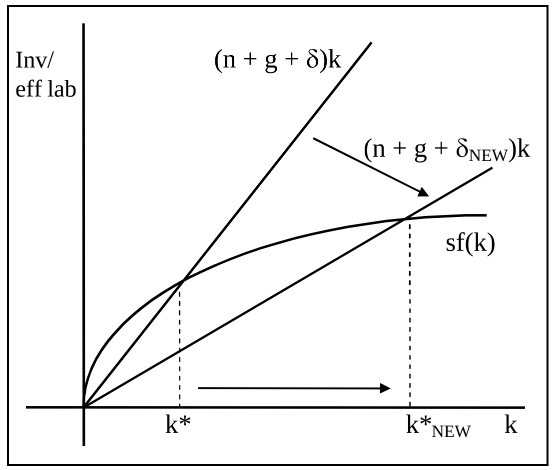


<u>Problem 1.3</u>

(a) The slope of the break-even investment line is given by (n + g + δ) and thus a fall in the rate of depreciation, δ, decreases the slope of the break-even investment line.

The actual investment curve, sf(k) is unaffected.

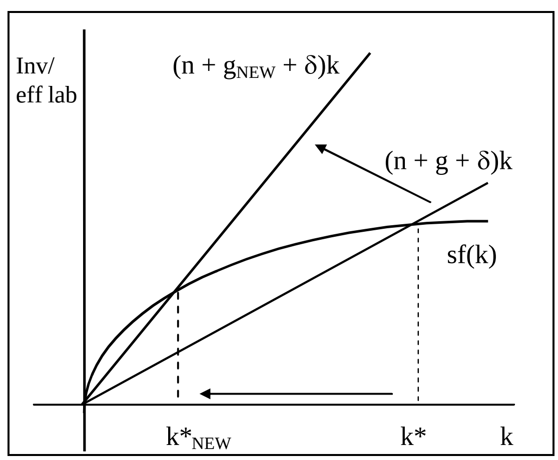
From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor rises from k* to k^*_{NEW} .



(b) Since the slope of the break-even investment line is given by (n + g + δ), a rise in the rate of technological progress, g, makes the break-even investment line steeper.

The actual investment curve, sf(k), is unaffected.

From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor falls from k* to k^*_{NEW} .



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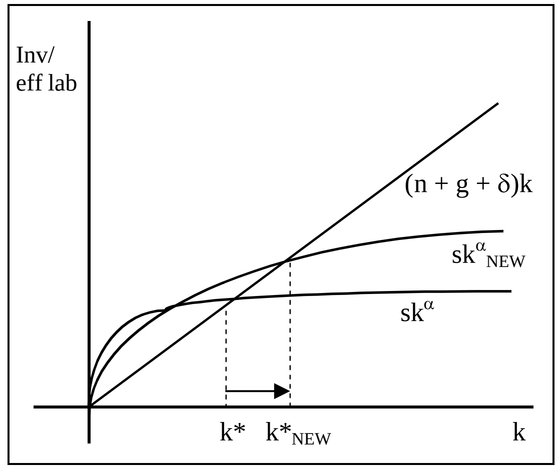


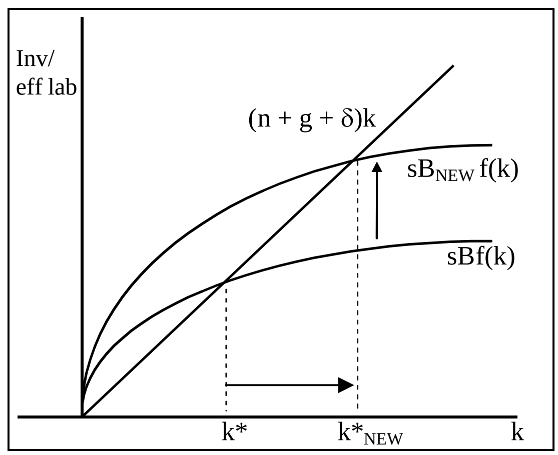
(c) The break-even investment line, $(n + g + \delta)k$, is unaffected by the rise in capital's share, α.

The effect of a change in α on the actual investment curve, sk^{α}, can be determined by examining the derivative $\partial(sk^{\alpha})/\partial\alpha$. It is possible to show that

(1) $$\frac{\partial sk^{\alpha}}{\partial\alpha} = sk^{\alpha}\ln k.$$

For $0 < \alpha < 1$, and for positive values of k, the sign of $\partial(sk^{\alpha})/\partial\alpha$ is determined by the sign of lnk. For lnk > 0, or k > 1, $\partial sk^{\alpha}/\partial\alpha > 0$ and so the new actual investment curve lies above the old one. For lnk < 0 or k < 1, $\partial sk^{\alpha}/\partial\alpha < 0$ and so the new actual investment curve lies below the old one. At k = 1, so that lnk = 0, the new actual investment curve intersects the old one.

In addition, the effect of a rise in α on k* is ambiguous and depends on the relative magnitudes of s and $(n + g + \delta)$. It is possible to show that a rise in capital's share, α, will cause k* to rise if $s > (n + g + \delta)$. This is the case depicted in the figure above.



(d) Suppose we modify the intensive form of the production function to include a non-negative constant, B, so that the actual investment curve is given by sBf(k), B > 0.

Then workers exerting more effort, so that output per unit of effective labor is higher than before, can be modeled as an increase in B. This increase in B shifts the actual investment curve up.

The break-even investment line, $(n + g + \delta)k$, is unaffected.

From the figure at right we can see that the balanced-growth-path level of capital per unit of effective labor rises from k* to k^*_{NEW}.

Problem 1.4

(a) At some time, call it t_0, there is a discrete upward jump in the number of workers. This reduces the amount of capital per unit of effective labor from k* to k_{NEW}. We can see this by simply looking at the definition, $k \equiv K/AL$. An increase in L without a jump in K or A causes k to fall. Since $f'(k) > 0$, this fall in the amount of capital per unit of effective labor reduces the amount of output per unit of effective labor as well. In the figure below, y falls from y* to y_{NEW}.

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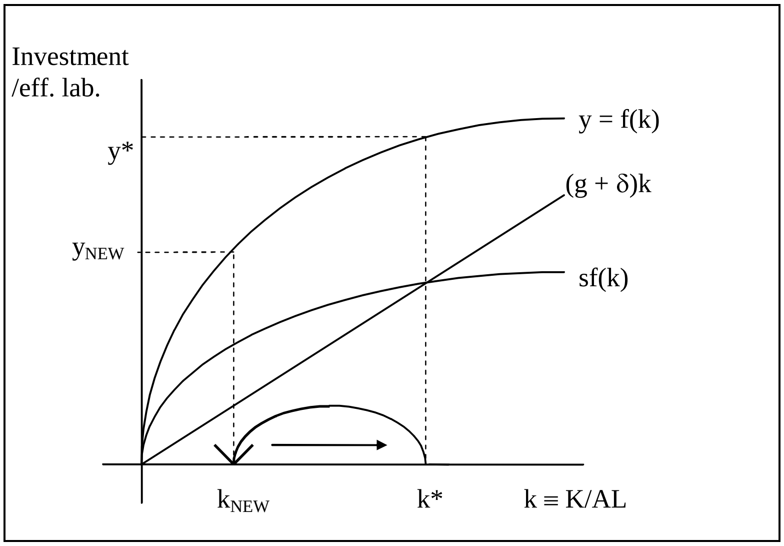
(b) Now at this lower k_{NEW}, actual investment per unit of effective labor exceeds break-even investment per unit of effective labor. That is, $sf(k_{NEW}) > (g + \delta)k_{NEW}$. The economy is now saving and investing more than enough to offset depreciation and technological progress at this lower k_{NEW}. Thus k begins rising back toward k*. As capital per unit of effective labor begins rising, so does output per unit of effective labor. That is, y begins rising from y_{NEW} back toward y*.



(c) Capital per unit of effective labor will continue to rise until it eventually returns to the original level of k*. At k*, investment per unit of effective labor is again just enough to offset technological progress and depreciation and keep k constant. Since k returns to its original value of k* once the economy again returns to a balanced growth path, output per unit of effective labor also returns to its original value of $y^* = f(k^*)$.

Problem 1.5

(a) The equation describing the evolution of the capital stock per unit of effective labor is given by

(1) $\dot{k} = sf(k) - (n + g + \delta)k$.

Substituting in for the intensive form of the Cobb-Douglas, $f(k) = k^{\alpha}$, yields

(2) $\dot{k} = sk^{\alpha} - (n + g + \delta)k$.

On the balanced growth path, $\dot{k}$ is zero; investment per unit of effective labor is equal to break-even investment per unit of effective labor and so k remains constant. Denoting the balanced-growth-path value of k as k*, we have $sk^{*\alpha} = (n + g + \delta)k^*$. Rearranging to solve for k* yields

(3) $k^* = \left[s/(n + g + \delta)\right]^{1/(1-\alpha)}$.

To get the balanced-growth-path value of output per unit of effective labor, substitute equation (3) into the intensive form of the production function, $y = k^{\alpha}$:

(4) $y^* = \left[s/(n + g + \delta)\right]^{\alpha/(1-\alpha)}$.

Consumption per unit of effective labor on the balanced growth path is given by $c^* = (1 - s)y^*$. Substituting equation (4) into this expression yields

(5) $c^* = (1 - s)\left[s/(n + g + \delta)\right]^{\alpha/(1-\alpha)}$.

(b) By definition, the golden-rule level of the capital stock is that level at which consumption per unit of effective labor is maximized. To derive this level of k, take equation (3), which expresses the balanced-growth-path level of k, and rearrange it to solve for s:

(6) $s = (n + g + \delta)k^{*1-\alpha}$.

Now substitute equation (6) into equation (5):

(7) $c^* = \left[1 - (n + g + \delta)k^{*1-\alpha}\right]\left[(n + g + \delta)k^{*1-\alpha}/(n + g + \delta)\right]^{\alpha/(1-\alpha)}$.

After some straightforward algebraic manipulation, this simplifies to

(8) $c^* = k^{*\alpha} - (n + g + \delta)k^*$.

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Equation (8) states that consumption per unit of effective labor is equal to output per unit of effective labor, $k^{*\alpha}$, less actual investment per unit of effective labor. On the balanced growth path, actual investment per unit of effective labor is the same as break-even investment per unit of effective labor, $(n + g + \delta)k^*$.

Now use equation (8) to maximize c* with respect to k*. The first-order condition is given by

(9) $\partial c^*/\partial k^* = \alpha k^{*\alpha-1} - (n + g + \delta) = 0$,

or simply

(10) $\alpha k^{*\alpha-1} = (n + g + \delta)$.

Note that equation (10) is just a specific form of the general condition that implicitly defines the golden-rule level of capital per unit of effective labor, given by $f'(k^*) = (n + g + \delta)$. Equation (10) has a graphical interpretation: it defines the level of k at which the slope of the intensive form of the production function is equal to the slope of the break-even investment line. Solving equation (10) for the golden-rule level of k yields

(11) $k^*_{GR} = \left[\alpha/(n + g + \delta)\right]^{1/(1-\alpha)}$.

(c) To get the saving rate that yields the golden-rule level of k, substitute equation (11) into (6):

(12) $s_{GR} = (n + g + \delta)\left[\alpha/(n + g + \delta)\right]^{(1-\alpha)/(1-\alpha)}$,

which simplifies to

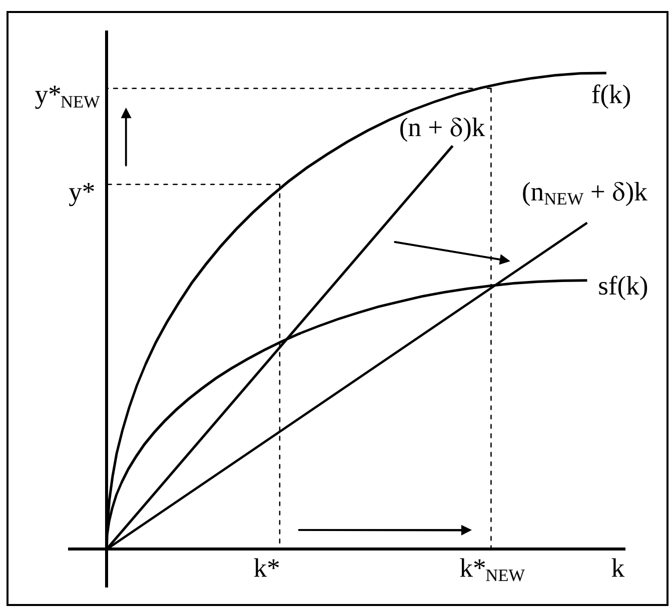
(13) $s_{GR} = \alpha$.

With a Cobb-Douglas production function, the saving rate required to reach the golden rule is equal to the elasticity of output with respect to capital or capital's share in output (if capital earns its marginal product).

Problem 1.6

(a) Since there is no technological progress, we can carry out the entire analysis in terms of capital and output per worker rather than capital and output per unit of effective labor. With A constant, they behave the same. Thus we can define $y \equiv Y/L$ and $k \equiv K/L$.

The fall in the population growth rate makes the break-even investment line flatter. In the absence of technological progress, the per unit time change in k, capital per worker, is given by $\dot{k} = sf(k) - (\delta + n)k$. Since $\dot{k}$ was 0 before the decrease in n – the economy was on a balanced growth path – the decrease in n causes $\dot{k}$ to become positive. At k*, actual investment per worker, sf(k*), now exceeds break-even investment per worker, $(n_{NEW} + \delta)k^*$. Thus k moves to a new higher balanced growth path level. See the figure at right.



As k rises, y – output per worker – also rises. Since a constant fraction of output is saved, c – consumption per worker – rises as y rises. This is summarized in the figures below.

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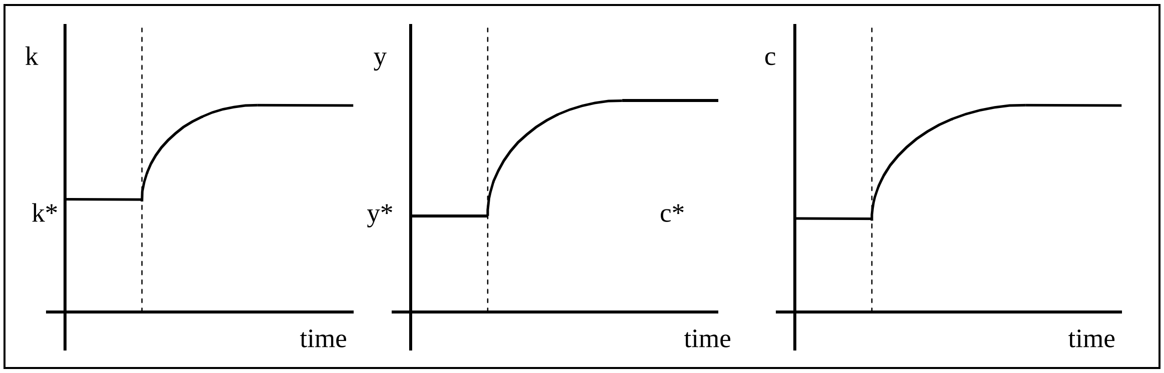
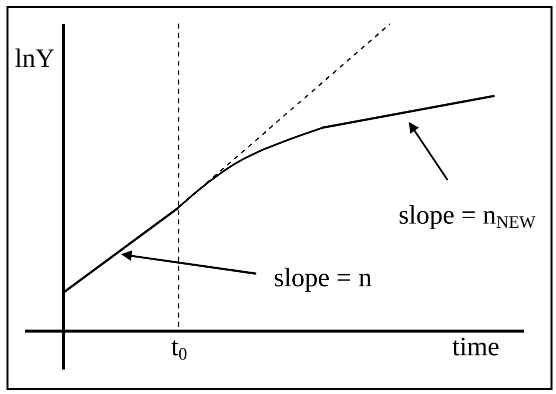


(b) By definition, output can be written as $Y \equiv Ly$. Thus the growth rate of output is $\dot{Y}/Y = \dot{L}/L + \dot{y}/y$. On the initial balanced growth path, $\dot{y}/y = 0$ – output per worker is constant – so $\dot{Y}/Y = \dot{L}/L = n$. On the final balanced growth path, $\dot{y}/y = 0$ again – output per worker is constant again – and so $\dot{Y}/Y = \dot{L}/L = n_{NEW} < n$. In the end, output will be growing at a permanently lower rate.



What happens during the transition? Examine the production function Y = F(K,AL). On the initial balanced growth path AL, K and thus Y are all growing at rate n. Then suddenly AL begins growing at some new lower rate n_{NEW}. Thus suddenly Y will be growing at some rate between that of K (which is growing at n) and that of AL (which is growing at n_{NEW}). Thus, during the transition, output grows more rapidly than it will on the new balanced growth path, but less rapidly than it would have without the decrease in population growth. As output growth gradually slows down during the transition, so does capital growth until finally K, AL, and thus Y are all growing at the new lower n_{NEW}.

Problem 1.7

The derivative of y* = f(k*) with respect to n is given by

(1) $\partial y^*/\partial n = f'(k^*)[\partial k^*/\partial n]$.

To find $\partial k^*/\partial n$, use the equation for the evolution of the capital stock per unit of effective labor, $\dot{k} = sf(k) - (n + g + \delta)k$. In addition, use the fact that on a balanced growth path, $\dot{k} = 0$, k = k* and thus sf(k*) = (n + g + δ)k*. Taking the derivative of both sides of this expression with respect to n yields

(2) $sf'(k^*)\dfrac{\partial k^*}{\partial n} = (n + g + \delta)\dfrac{\partial k^*}{\partial n} + k^*$,

and rearranging yields

(3) $\dfrac{\partial k^*}{\partial n} = \dfrac{k^*}{sf'(k^*) - (n + g + \delta)}$.

Substituting equation (3) into equation (1) gives us

(4) $\dfrac{\partial y^*}{\partial n} = f'(k^*)\left[\dfrac{k^*}{sf'(k^*) - (n + g + \delta)}\right]$.

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Rearranging the condition that implicitly defines k*, sf(k*) = (n + g + δ)k*, and solving for s yields
(5) $s = (n + g + \delta)k^*/f(k^*)$.
Substitute equation (5) into equation (4):

(6) $$\frac{\partial y^*}{\partial n} = \frac{f'(k^*)k^*}{[(n+g+\delta)f'(k^*)k^*/f(k^*)] - (n+g+\delta)}.$$

To turn this into the elasticity that we want, multiply both sides of equation (6) by n/y*:

(7) $$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = \frac{n}{(n+g+\delta)}\frac{f'(k^*)k^*/f(k^*)}{[f'(k^*)k^*/f(k^*)]-1}.$$

Using the definition that $\alpha_K(k^*) \equiv f'(k^*)k^*/f(k^*)$ gives us

(8) $$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = -\frac{n}{(n+g+\delta)}\left[\frac{\alpha_K(k^*)}{1-\alpha_K(k^*)}\right].$$

Now, with $\alpha_K(k^*) = 1/3$, g = 2% and δ = 3%, we need to calculate the effect on y* of a fall in n from 2% to 1%. Using the midpoint of n = 0.015 to calculate the elasticity gives us

(9) $$\frac{n}{y^*}\frac{\partial y^*}{\partial n} = -\frac{0.015}{(0.015+0.02+0.03)}\left(\frac{1/3}{1-1/3}\right) \cong -0.12.$$

So this 50% drop in the population growth rate, from 2% to 1%, will lead to approximately a 6% increase in the level of output per unit of effective labor, since (-0.50)(-0.12) = 0.06. This calculation illustrates the point that observed differences in population growth rates across countries are not nearly enough to account for differences in y that we see.

Problem 1.8

(a) A permanent increase in the fraction of output that is devoted to investment from 0.15 to 0.18 represents a 20 percent increase in the saving rate. From equation (1.27) in the text, the elasticity of output with respect to the saving rate is

(1) $$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)},$$

where $\alpha_K(k^*)$ is the share of income paid to capital (assuming that capital is paid its marginal product).

Substituting the assumption that $\alpha_K(k^*) = 1/3$ into equation (1) gives us

(2) $$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)} = \frac{1/3}{1-1/3} = \frac{1}{2}.$$

Thus the elasticity of output with respect to the saving rate is 1/2. So this 20 percent increase in the saving rate – from s = 0.15 to $s_{NEW} = 0.18$ – causes output to rise relative to what it would have been by about 10 percent. [Note that the analysis has been carried out in terms of output per unit of effective labor. Since the paths of A and L are not affected, however, if output per unit of effective labor rises by 10 percent, output itself is also 10 percent higher than what it would have been.]

(b) Consumption rises less than output. Output ends up 10 percent higher than what it would have been. But the fact that the saving rate is higher means that we are now consuming a smaller fraction of output. We can calculate the elasticity of consumption with respect to the saving rate. On the balanced growth path, consumption is given by
(3) $c^* = (1 - s)y^*$.
Taking the derivative with respect to s yields

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(4) $\frac{\partial c^*}{\partial s} = -y^* + (1-s)\frac{\partial y^*}{\partial s}$.

To turn this into an elasticity, multiply both sides of equation (4) by s/c*:

(5) $\frac{\partial c^*}{\partial s}\frac{s}{c^*} = \frac{-y^* s}{(1-s)y^*} + (1-s)\frac{\partial y^*}{\partial s}\frac{s}{(1-s)y^*}$,

where we have substituted c* = (1 - s)y* on the right-hand side. Simplifying gives us

(6) $\frac{\partial c^*}{\partial s}\frac{s}{c^*} = \frac{-s}{(1-s)} + \frac{\partial y^*}{\partial s}\frac{s}{(1-s)y^*}$.

From part (a), the second term on the right-hand side of (6), the elasticity of output with respect to the saving rate, equals 1/2. We can use the midpoint between s = 0.15 and s_{NEW} = 0.18 to calculate the elasticity:

(7) $\frac{\partial c^*}{\partial s}\frac{s}{c^*} = \frac{-0.165}{(1-0.165)} + 0.5 \cong 0.30$.

Thus the elasticity of consumption with respect to the saving rate is approximately 0.3. So this 20% increase in the saving rate will cause consumption to be approximately 6% above what it would have been.

(c) The immediate effect of the rise in investment as a fraction of output is that consumption falls. Although y* does not jump immediately – it only begins to move toward its new, higher balanced-growth-path level – we are now saving a greater fraction, and thus consuming a smaller fraction, of this same y*. At the moment of the rise in s by 3 percentage points – since c = (1 - s)y* and y* is unchanged – c falls. In fact, the percentage change in c will be the percentage change in (1 - s). Now, (1 - s) falls from 0.85 to 0.82, which is approximately a 3.5 percent drop. Thus at the moment of the rise in s, consumption falls by about three and a half percent.

We can use some results from the text on the speed of convergence to determine the length of time it takes for consumption to return to what it would have been without the increase in the saving rate. After the initial rise in s, s remains constant throughout. Since c = (1 - s)y, this means that consumption will grow at the same rate as y on the way to the new balanced growth path. In the text it is shown that the rate of convergence of k and y, after a linear approximation, is given by $\lambda = (1 - \alpha_K)(n + g + \delta)$. With $(n + g + \delta)$ equal to 6 percent per year and $\alpha_K = 1/3$, this yields a value for λ of about 4 percent. This means that k and y move about 4 percent of the remaining distance toward their balanced-growth-path values of k* and y* each year. Since c is proportional to y, c = (1 - s)y, it also approaches its new balanced-growth-path value at that same constant rate. That is, analogous to equation (1.31) in the text, we could write

(8) $c(t) - c^* \cong e^{-(1-\alpha_K)(n+g+\delta)t}[c(0) - c^*]$,

or equivalently

(9) $e^{-\lambda t} = \frac{c(t) - c^*}{c(0) - c^*}$.

The term on the right-hand side of equation (9) is the fraction of the distance to the balanced growth path that remains to be traveled.

We know that consumption falls initially by 3.5 percent and will eventually be 6 percent higher than it would have been. Thus it must change by 9.5 percent on the way to the balanced growth path. It will therefore be equal to what it would have been about 36.8 percent (3.5%/9.5% ≅ 36.8%) of the way to the new balanced growth path. Equivalently, this is when the remaining distance to the new balanced growth

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